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# FURTHER CHARACTERIZATIONS OF FUNCTIONS OF A PAIR OF ORTHOGONAL PROJECTORS

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**Abstract**

The paper provides several original conditions involving ranks and traces of functions of a pair of orthogonal projectors (i.e., Hermitian idempotent matrices) under which the functions themselves are orthogonal projectors. The results are established by means of a joint decomposition of the two projectors.

**Keywords:** Hermitian idempotent matrix, partial isometry, rank, trace, Moore-Penrose inverse.

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## APPENDIX

In what follows we provide the representations of the Moore–Penrose inverses of selected functions of orthogonal projectors $\mathbf{P}$ and $\mathbf{Q}$ having the forms (**??**) and (**??**), respectively.

$$(\mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{B^*A^\dagger} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}+\mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \frac{1}{2}\overline{\mathbf{P}}_{\overline{\mathbf{A}}} & -\mathbf{BD}^\dagger \\ -\mathbf{D}^\dagger\mathbf{B}^* & 2\mathbf{D}^\dagger - \mathbf{P_D} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}-\mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & -\mathbf{BD}^\dagger \\ -\mathbf{D}^\dagger\mathbf{B}^* & -\mathbf{P_D} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQP})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{A}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{I}_n - \mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQ}+\mathbf{QP})^\dagger = \mathbf{U}\begin{pmatrix} \frac{1}{2}\mathbf{A}^\dagger - \frac{1}{2}\mathbf{A}^\dagger\mathbf{B}(\mathbf{B}^*\mathbf{A}^\dagger\mathbf{B})^\dagger\mathbf{B}^*\mathbf{A}^\dagger & \mathbf{A}^\dagger\mathbf{B}(\mathbf{B}^*\mathbf{A}^\dagger\mathbf{B})^\dagger \\ (\mathbf{B}^*\mathbf{A}^\dagger\mathbf{B})^\dagger\mathbf{B}^*\mathbf{A}^\dagger & -2(\mathbf{B}^*\mathbf{A}^\dagger\mathbf{B})^\dagger \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQ}-\mathbf{QP})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{0} & -(\mathbf{B}^*)^\dagger \\ \mathbf{B}^\dagger & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{I}_n - \mathbf{P} - \mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} -\mathbf{P_A} & -\mathbf{A}^\dagger\mathbf{B} \\ -\mathbf{B}^*\mathbf{A}^\dagger & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ -\mathbf{D}^\dagger\mathbf{B}^* & \mathbf{D}^\dagger \end{pmatrix}\mathbf{U}^*.$$

Validity of these representations can be verified by exploiting the four Penrose conditions given in (**??**). Details on how most of these representations were derived can be found in articles [3, 4] and [6, 7].