Discussiones Mathematicae
Probability and Statistics 37 (2017) 101–122
doi:10.7151/dmps.1190

GAUSSIAN MIXTURES AND FINANCIAL RETURNS

C. Cuevas-Covarrubias, J. Iñigo-Martínez

and

R. Jímenez-Padilla

Centro de Investigación en Estadística y Matemáticas Aplicadas
Facultad de Ciencias Actuariales
Universidad Anáhuac México

e-mail: ccuevas@anahuac.mx

Abstract

Many important models in quantitative finance are based on the assumption that stock returns are independent and normally distributed. However, the empirical distributions of price changes are frequently skewed and leptokurtic. Therefore, flexible distributions and their potential in financial modeling constitute an important research topic in mathematical finance. We explore the potential of Gaussian mixtures as an alternative to the normal distribution. Our discussion is based on three practical examples taken from the Mexican stock market. This article is not limited to the estimation of marginal distributions. Contrasting with some other papers in the literature, the application of multivariate Gaussian mixtures to estimate joint distributions of financial returns is also analyzed. This multivariate approach gives us the opportunity to illustrate the application of Gaussian mixtures in portfolio theory and risk assessment.

Keywords: Gaussian mixtures, financial returns, VaR, CVaR.

2010 Mathematics Subject Classification: 62-09, 62P05, 62F10, 91G70.

1. Introduction

The normality of stock returns is a fundamental assumption in mathematical finance. However, empirical evidence often contradicts this theoretical foundation. As a consequence, the application of alternative models is an important research topic in Risk Theory (Jondeau [9], Klugman *et al.* [11]). This article presents empirical evidence suggesting that Gaussian mixtures (GM) constitute an effective tool for financial modeling. We can describe its content as a practical discussion based on three examples from the Mexican Stock Market[1]. Previous articles have also discussed the potential of GM in portfolio selection and risk assessment. However, most of these approaches concentrate on the estimation of marginal distributions and ignore the covariance structure among assets. Thus, the main contribution of our proposal is the application of the same ideas, but from a multivariate perspective. In other words, we take stochastic dependence into account, and we recognize its importance as a real source of volatility and risk. The results are encouraging and invite us to reconsider the Normal distribution as a valid foundation in Risk Theory and Financial Modeling. In order to make the practical potential of mixture models more evident, we illustrate the estimation of joint distributions of financial returns, and exploit the resulting models to minimize the maximum expected loss of a portfolio, where VaR and CVaR are easily calculated thanks to the properties of GM.

The application of copula functions is a standard in financial practice; therefore, we use it here as a benchmark to assess the performance of our multivariate GM approach. We conclude that GM offer a powerful family of flexible distributions that deserve to be studied from a new perspective in Risk Management and Quantitative Finance.

2. The normal distribution in finance

Stock prices are frequently assumed to follow a geometric Brownian motion process; therefore, financial returns are considered to be independent and normally distributed. In their seminal article, Black and Scholes [2] assume that "*... the stock price follows a random walk in continuous time ... Thus the distribution of possible stock prices at the end of any finite interval is log-normal.*" The normality of returns is a fundamental hypothesis in financial mathematics. However, stock returns often show skewed and leptokurtic empirical distributions that contradict this distributional assumption. The non Gaussianity of price movements is a well known fact since the 1960's; nevertheless, it still motivates an interesting discussion in Quantitative Finance (for instance: Mota [18] and Esch [4]). In 1965 Fama [5] explains how the normal distribution hypothesis was not seriously questioned until the work of Benoit Mandelbrot [12]. According to Fama [5], Mandelbrot's main assertion is that academic research had "*readily neglected the implications of leptokurtosis usually observed in empirical distributions of price changes*".

[1] *Bolsa Mexicana de Valores*

An increasing interest for Gaussian mixtures and its possible applications in finance has been shown in the literature during the last two decades. Zhang and Cheng [21] propose an original criterion to calculate the VaR based on the assumption of Gaussian mixture returns. In the Markowitz mean-variance portfolio framework, the normality assumption of asset returns is necessary to "easily" solve the optimization problem. In [3], Buckley *et al.* optimized —by changing the objective function and the constraints of the Quadratic Programming problem— the portfolio to find the weights of each asset when they follow a Gaussian Mixture distribution. Buckley *et al.* [3] write: *"the new approach is ideal for an industrial setting, providing considerable additional flexibility over and above a standard Markowitz approach, with only a modest increase in complexity"*. Buckley *et al.* also discuss three limitations of the multivariate normal assumption for joint asset returns: skewness, leptokurtosis and asymmetric correlation. Esch [4] presents a discussion about the application of flexible non Gaussian distributions in finance. He criticises the summary rejection of the normal distribution and recommends Gaussian mixtures as an alternative to model the returns of assets. In a more recent article, Tan and Chu [20] explore the application of Gaussian mixture models in portfolio selection and some of its theoretical implications in the calculation of the VaR.

Recently, several articles have reported empirical evidence supporting the application of Gaussian mixtures in Quantitative Finance. Behr and Poetter [1] analyze the marginal distribution of the returns of ten European stock market indexes. They compare different flexible models and they conclude that Gaussian mixtures seem to be slightly superior than the others. In a similar way, Kamaruzzaman *et al.* [10] apply GM models to estimate the distribution of the returns of three stock market indexes in Malaysia. This paper presents empirical evidence supporting the application of Gaussian mixtures in financial modeling. It is based on three series from the Mexican Stock Market. The results are interesting and suggest that GM are of great practical value.

3. Finite Gaussian mixtures

Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let f be its corresponding density function. We say that X is distributed according to a Gaussian Mixture when

$$f(x) = \sum_{j=1}^{k} \pi_j \frac{1}{\sigma_j} \phi\Big(\frac{x - \mu_j}{\sigma_j}\Big). \tag{1}$$

In the previous equation, ϕ represents the density of a standard normal and $\pi_1, \pi_2, \dots, \pi_k$ are positive constants such that $\sum_{j=1}^{k} \pi_j = 1$. Then, it is possible to assume that our sample space Ω is such that $\Omega = \bigcup_{j=1}^{k} \Omega_j$ and $\Omega_i \Omega_j = \emptyset$ $\forall\, i \neq j$. Where each π_j in equation (1) represents the *prior* probability for Ω_j and each $\frac{1}{\sigma_j}\phi\big(\frac{x-\mu_2}{\sigma_2}\big)$ is the conditional probability density of X given Ω_j. Each Ω_j represents a specific *regime* of financial returns. Given x, a particular realization of X, the *posterior* probabilities for each regime Ω_j are given by

$$\Pr[\Omega_j | X = x] = \frac{\frac{\pi_j}{\sigma_j}\phi\big(\frac{x-\mu_j}{\sigma_j}\big)}{\sum_{h=1}^{k} \frac{\pi_h}{\sigma_h}\phi\big(\frac{x-\mu_h}{\sigma_h}\big)}; \quad j = 1, 2, \ldots, k. \tag{2}$$

GM models are flexible. Gridgeman [7] proofs that "*a mixture of different normal distributions with a common mean is leptokurtic*". An original proof of this proposal is presented in the appendix at the end of this article. Given their capacity to model leptokurtic densities, GM models constitute a natural alternative to model the distribution of financial returns. Figure 1 illustrates the flexibility of GM models. It compares a standard Normal density to three different versions of the following GM:

$$f(x) = \pi \frac{1}{\sigma_0}\phi\Big(\frac{x}{\sigma_0}\Big) + (1-\pi)\frac{1}{\sigma_1}\phi\Big(\frac{x}{\sigma_1}\Big). \tag{3}$$

The first element of the three GM models shown in Figure 1 is a normal density with $\sigma_0 = 0.5$. Model 1 has a weight parameter $\pi = 0.75$, and its second element is a Normal density with standard deviation $\sigma_1 = 1.17$. The second element of Model 2 is a normal density with $\sigma_1 = 1.50$ and its weight parameter is $\pi = 0.50$. Finally, the parameter values for Model 3 are: $\sigma_1 = 10.5$ and $\pi = 0.05$. All GM in Figure 1 are centered at the the origin and have unit variance; however, all have different levels of kurotosis.

GM models inherit many convenient properties from their normal components (Tan and Chu [20]). This makes them attractive and comfortable to work with. Mixtures of normals may be defined in terms of two, three or even or more parameters. Thus, according to what we need, these models may be flexible or parsimonious. Depending on the value of its parameters, a mixture of $k \geq 2$ normal densities may be unimodal, symmetric, skewed, multimodal, etc. (See McLachlan and Peel [17] and Fruhwirth-Schnatter [6]). Given their great flexibility, GM constitute a powerful tool in financial modeling.

GM are not limited to a univariate context. A random vector $X : \Omega \to \mathbb{R}^p$ is distributed according to a Multivariate Gaussian Mixture if its density function is a convex linear combination of multivariate normal densities. This is:

$$f(x) = \sum_{j=1}^{k} \pi_j \phi\big(x | \mu_j, \Sigma_j\big). \tag{4}$$

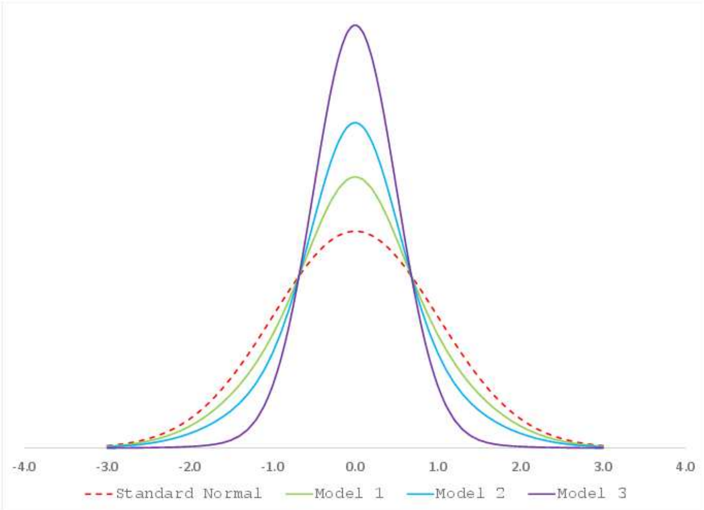

Figure 1. Three leptokurtic Gaussian mixtures and a standard normal.

where

$$\phi(x|\mu_j,\Sigma_j)=\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma_j|^{\frac{1}{2}}}\exp\Big[-\frac{1}{2}(x-\mu_j)^t\Sigma_j^{-1}((x-\mu_j))\Big] \tag{5}$$

is the density function of a multivariate normal centered at μ_j and with covariance matrix Σ_j (see Mardia *et al.* [13], p. 59).

4. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be an observed random sample coming from a univariate GM like the one in equation (1) and let

$$\theta=\big(\pi_1,\pi_2,\ldots,\pi_k,\mu_1,\mu_2,\ldots,\mu_k,\sigma_1^2,\sigma_2^2,\ldots,\sigma_k^2\big)^t$$

be its vector of parameters. Then, its corresponding log-likelihood function is:

$$\mathcal{L}(\theta)=\sum_{i=1}^{n}\ln\Bigg(\sum_{j=1}^{k}\frac{\pi_j}{\sigma_j}\phi\Big(\frac{x-\mu_j}{\sigma_j}\Big)\Bigg). \tag{6}$$

In order to estimate θ, $\mathcal{L}$ must be maximized subject to $\sum_{j=1}^{k}\pi_j=1$. After applying Lagrange multipliers, this problem translates into the following system of equations:

$$
(7) \qquad \begin{cases} \frac{1}{n}\sum_{i=1}^{n}\Pr[\Omega_j|X=x_i] = \pi_j, & \text{for } j=1\ldots k \\ \sum_{i=1}^{n}\Pr[\Omega_j|X=x_i]\frac{\partial}{\partial\mu_j}\ln\left(\frac{1}{\sigma_j}\phi\left(\frac{x_i-\mu_j}{\sigma_j}\right)\right) = 0, & \text{for } j=1\ldots k \\ \sum_{i=1}^{n}\Pr[\Omega_j|X=x_i]\frac{\partial}{\partial\sigma_j^2}\ln\left(\frac{1}{\sigma_j}\phi\left(\frac{x_i-\mu_j}{\sigma_j}\right)\right) = 0, & \text{for } j=1\ldots k. \end{cases}
$$

The posteriors in equation (7) are defined as in equation (2). Therefore, the maximum likelihood estimate of each π_j results to be the arithmetic mean of the posteriors for its corresponding Ω_j and for each observation x_i in the sample. The left hand side of the last $2k$ equations in (7), can be interpreted as the first derivatives of a *weighted log-likelihood*: the larger the value of $\Pr[\Omega_j|X=x_i]$, the larger the contribution of x_i to the estimation of μ_j and σ_j^2. After some algebraic work, the last $2k$ equations in (7) can be transformed into

$$
(8) \qquad \begin{cases} \hat{\mu}_j = \sum_{i=1}^{n} W_{i,j}x_i, \text{ for } j=1\ldots k, \\ \hat{\sigma}_j^2 = \sum_{i=1}^{n} W_{i,j}(x_i-\hat{\mu}_j)^2, \text{ for } j=1\ldots k, \end{cases}
$$

where $W_{i,j} = \frac{\Pr[\Omega_j|X=x_i]}{\sum_{h=1}^{n}\Pr[\Omega_j|X=x_h]}$.

The Maximum Likelihood estimate of θ can be obtained by the following iterative process known as the EM Algorithm:

EM-Algorithm:

1. Define initial values for the $3k-1$ parameters in the mixture.
2. Calculate $\Pr[\Omega_j|X=x_i]$ for $i=1,2,\ldots,n$ and for $j=1,2,\ldots,k$ according to equation (2).
3. **Expectation Step:** Compute the priors $\pi_1,\pi_2,\ldots,\pi_k$ according to equation (7).
4. **Maximization Step:** Estimate values for $\hat{\mu}_1,\hat{\mu}_2,\ldots,\hat{\mu}_k$ and for $\hat{\sigma}_1^2,\hat{\sigma}_2^2,\ldots,\hat{\sigma}_k^2$ according to equation (8).
5. Repeat the cycle from step 2 until convergence is reached.

The EM-Algorithm is illustrated by the flow chart shown in Figure 2. It is reliable and easy to implement. An interesting discussion about its theoretical foundation is given in McLachlan and Peel [17] and McLachlan *et al.* [15]. Its practical application is illustrated by the examples shown in the following sections.

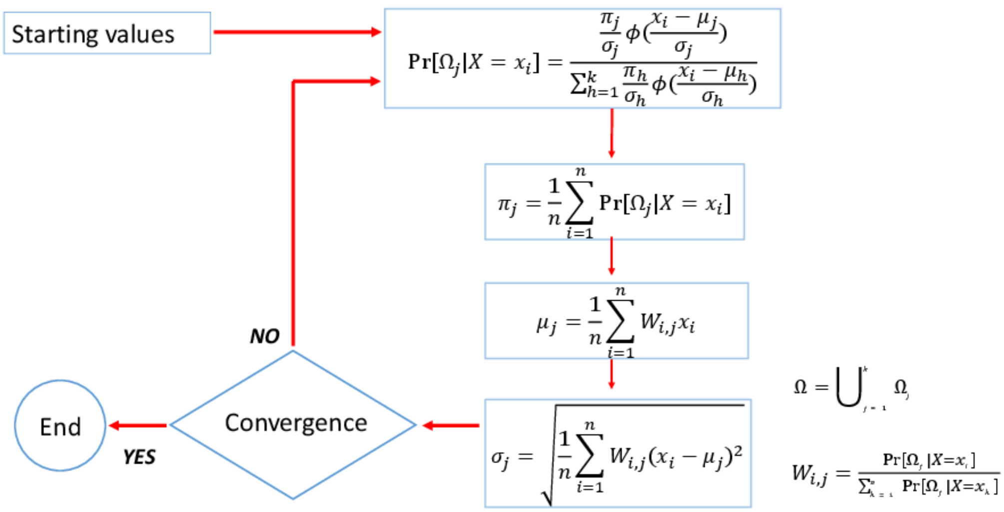

Figure 2. EM algorithm for a univariate Gaussian mixture.

5. UNIVARIATE MODELING FOR MARGINAL RETURNS

Example 1. Mexican Stock Market Index; MEXBOL IPC.

We analyze one thousand daily log-returns of the Mexican stock index (Bolsa IPC) from June 30^{th} 2008 to June 18^{th} 2012. The sample series and its corresponding histogram are shown in Figure 3. Our objective is to estimate its distribution function. In a first step, we use maximum likelihood to fit a single normal model and we asses its goodness of fit with a Kolmogorov-Smirnov test. The results are shown in Table 1. Figure 4 compares the resulting normal distribution with its empirical counterpart; clearly, the normal model has an important lack of fit.

μ	σ	K-S	
0.00026	0.015	0.09	pv $\leq$ 0.01

Table 1. IPC daily log returns; normal distribution fit.

The series of daily returns of the IPC index at the left hand side of Figure 3 shows sudden contractions and expansions of volatility, this volatility clusters are a common feature in random sequences generated by certain forms of GM. The histogram at the rigth hand side has a peaked shape that also suggests the

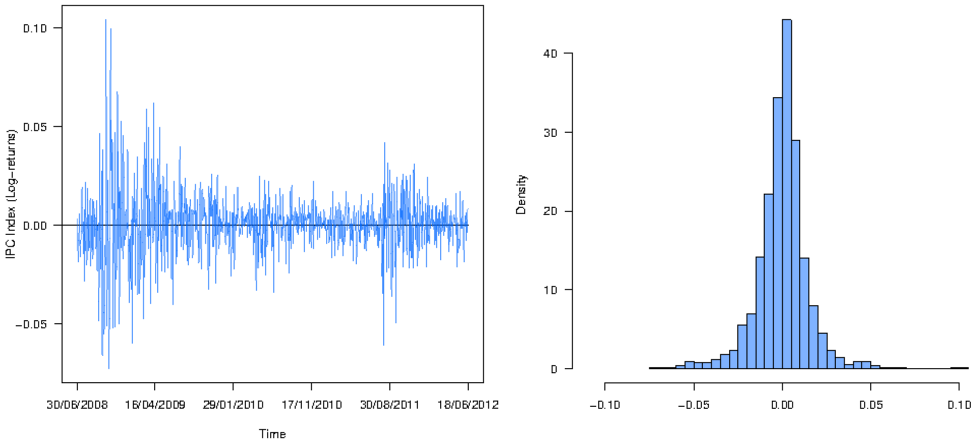

Figure 3. Bolsa IPC Index daily log-returns.

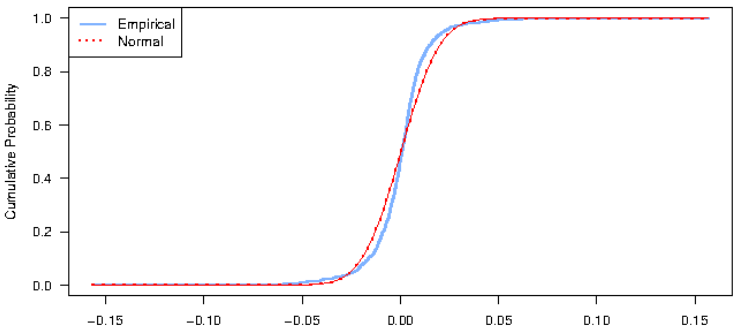

Figure 4. Bolsa IPC Index daily log-Returns; normal distribution fit.

possibility of a mixture of normals. Thus, we decided to fit the following model:

$$f(x) = \pi \frac{1}{\sigma_1} \phi\Big(\frac{x-\mu_1}{\sigma_1}\Big) + (1-\pi)\frac{1}{\sigma_2}\phi\Big(\frac{x-\mu_2}{\sigma_2}\Big).$$

Maximum Likelihood was applied. Table 2 shows the point estimates for each parameter in the mixture. It is interesting to see how the first element of the mixture (regime Ω_1) has a negative expected return and a relatively large volatility. This distribution describes the behavior of daily returns in periods of financial stress. According to the model, this difficult regime was observed 25% of the time.

The Kolmogorov-Smirnov test does not show any evidence against this GM model.

	Regime Ω_1		Regime Ω_2			
π	μ_1	σ_1	μ_2	σ_2	K-S	
0.2457	-0.001613	0.02764	0.000872	0.00836	0.02	pv > 0.1

Table 2. IPC daily log returns. Gaussian mixture fit.

Figure 5 compares it with the empirical distribution. Additionally, Figure 6 compares an empirical density estimate based on kernel smoothing with the normal and the GM densities; the mixture model seems to offer an excellent fit.

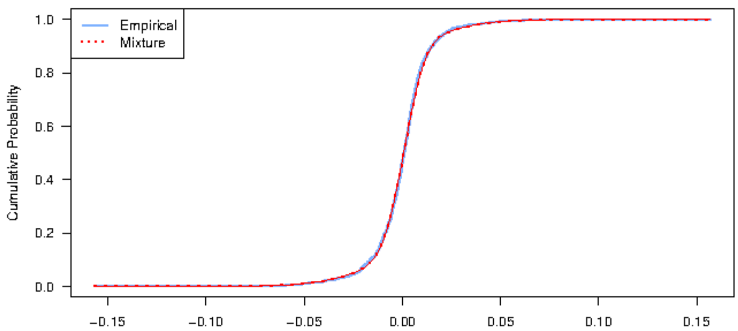

Figure 5. Bolsa IPC Index daily log-Returns; Gaussian mixture fit.

Example 2. TELMEX L AND AMX L.

Now we analyze the marginal log-returns of two assets from the Mexican stock market: TELMEX L and AMX L. We study their behavior from march 4^{th}, 2008 to april 15^{th}, 2010 (a total of 529 observations). The Kolmogorov-Smirnov test rejected the normal distribution in both cases. The results of this analysis are shown in Table 3 and Figure 7.

Asset	μ	σ	K-S	
TELMEX L	-0.00005	0.0259	0.08	pv ≤ 0.01
AMX L	0.00008	0.0196	0.07	pv ≤ 0.01

Table 3. TELMEX L and AMX L daily log returns. Normal distribution fit.

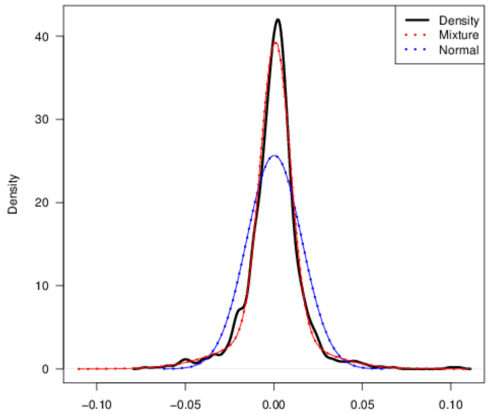

Figure 6. Bolsa IPC Index daily log-Returns; density estimation.

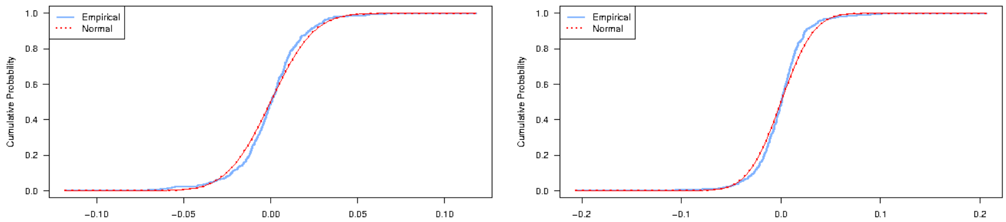

Figure 7. TELMEX L and AMX L marginal normal distribution fit.

Figures 8 and 9 show a graphical description of both series and suggest the possibility of a GM process. Thus, the EM-Algorithm was applied to fit the parsimonious model given in equation (3). The results are shown in Table 4 and Figure 10. Even though this model is defined in terms of three parameters only; the improvement with respect to the normal distribution is remarkable. In both cases the GM completely overlaps with its corresponding empirical distribution. In the next section, these two marginal densities will be linked through a Gaussian copula in order to estimate the joint distribution of both daily log-returns.

Asset	π	σ_1	σ_2	K-S	
TELMEX L	0.29	0.0308	0.0123	0.02	pv > 0.1
AMX L	0.18	0.0491	0.0169	0.02	pv > 0.1

Table 4. TELMEX L and AMX L daily log returns. Gaussian mixture fit.

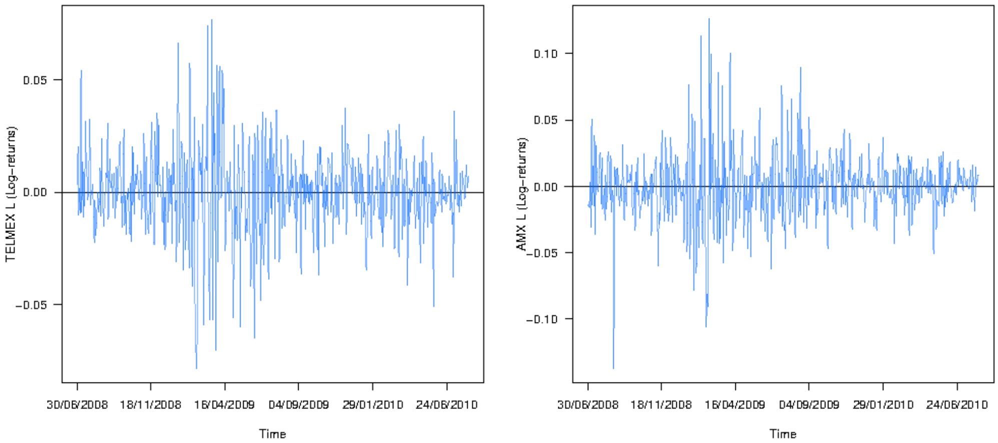

Figure 8. TELMEX L and AMX L marginal daily log-returns.

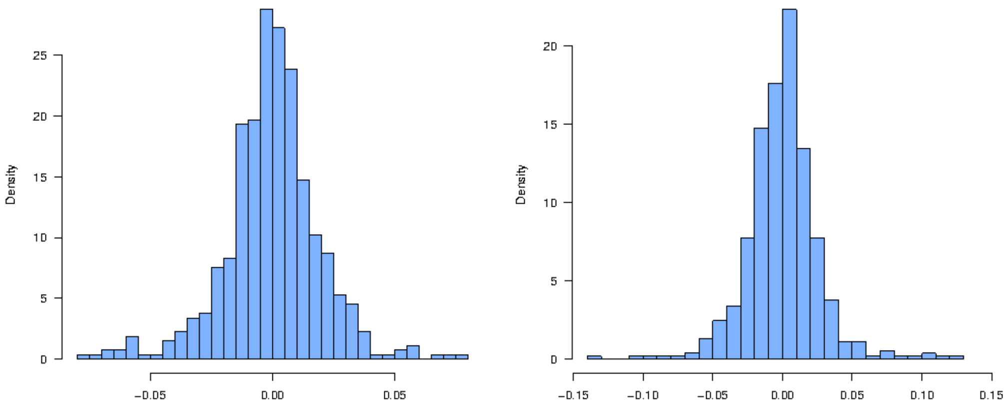

Figure 9. Marginal histograms; (left) TELMEX L, (right) AMX L.

6. BIVARIATE MODELING OF JOINT RETURNS

The previous section shows the potential of Gaussian mixture models to estimate univariate distribution functions in finance. We now illustrate their application in a multivariate context. Example 2 in Section 3 analyzes a series of 529 log-

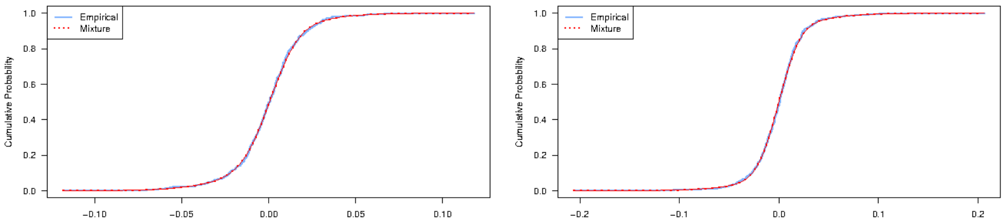

Figure 10. Marginal distributions; (left) TELMEX L, (right) AMX L.

returns of AMX L and TELMEX L. These securities were analyzed separately and their stochastic dependence was not taken into account. Now, both returns are considered to be dependent elements of a bivariate random vector, and maximum likelihood is applied in order to estimate their joint distribution. We follow two different approaches: one assumes this density to be a mixture of two bivariate normals; the other fits a Gaussian copula with Gaussian mixture marginals. The results are compared in terms of a goodness of fit assessment.

The Gaussian Mixture approach

The scatter plot in the left hand side of Figure 12 shows the observed sample of a random vector $\mathbf{V} = (X, Y)^t$: where X and Y represent the observed daily log-returns of TELMEX L and AMX L respectively. The shape of the scatter suggests that the density function of $\mathbf{V}$ is unimodal and that X and Y have a positive correlation.

We assume that $\mathbf{V}$ is distributed according to a multivariate Gaussian mixture as defined in equations (4) and (5) of Section 3. In this particular case $\mathbf{V}$ is bidimensional, therefore equation (5) is given in terms of five parameters and may be written as in equation (9).

(9)
$$\phi_j(x, y \mid \rho) = \frac{1}{2\pi\sigma_{j,x}\sigma_{j,y}\sqrt{1-\rho_j^2}} e^{-\frac{1}{2\left(1-\rho_j^2\right)}\left[\left(\frac{x-\mu_{j,x}}{\sigma_{j,x}}\right)^2 - 2\rho_j\left(\frac{x-\mu_{j,x}}{\sigma_{j,x}}\right)\left(\frac{y-\mu_{j,y}}{\sigma_{j,y}}\right) + \left(\frac{y-\mu_{j,y}}{\sigma_{j,y}}\right)^2\right]}$$

Each ρ_j is the conditional correlation between X and Y given regime Ω_j. In a similar way, $\mu_{j,x}$, $\mu_{j,y}$, $\sigma_{j,x}$ and $\sigma_{j,y}$ are their respective conditional expectations and standard deviations.

We applied the EM-algorithm to fit a mixture of two bivariate normals, the results are given in Table 5.

Regime	π_j	$\mu_{j,x}$	$\mu_{j,y}$	$\sigma_{j,x}$	$\sigma_{j,y}$	ρ_j
Ω_1	0.7644	0.0008	0.0004	0.0123	0.0160	0.4159
Ω_2	0.2358	0.0030	0.0014	0.0326	0.0449	0.5725

Table 5. TELMEX L and AMX L joint log returns. Gaussian mixture fit.

It is interesting to see how this mixture model describes the joint behavior of both assets (AMX L and TELMEX L) in terms of two different bivariate normals. The first component of the mixture (regime Ω_1) corresponds to a regular market behavior with low risk but moderate expected returns; this regime is observed 76% of the time. The second element of the mixture (regime Ω_2) describes a stressed environment with higher volatility and stronger dependence among both assets. According to the GM model, this stressed regime is observed 24% of the time. Even though Ω_2 implies a higher risk, it also has a larger expected return. Figure 11 shows the curves of level for this GM model and compares them with the curves of level of an empirical density function estimated by kernel smoothing. The GM model offers a good description of the joint behavior of these two assets.

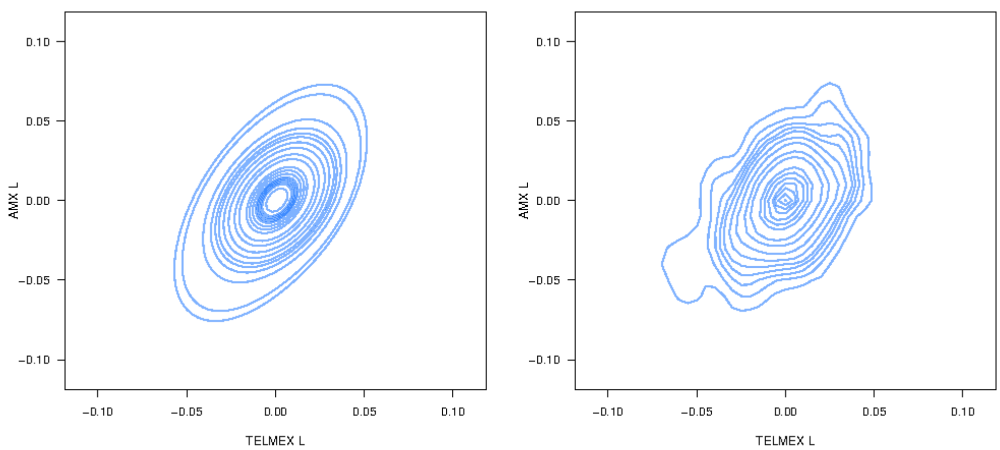

Figure 11. Curves of level: (L) Gaussian mixture, (R) Smooth empirical density.

In a similar way, Figure 12 compares the scatter plot of 529 random points generated from our GM model to our observed sample of TELMEX L and AMX L; both scatters are quite similar. We can conclude from this graphical assessment that the GM model offers a good estimation of the joint distribution of TELMEX L and AMX L.

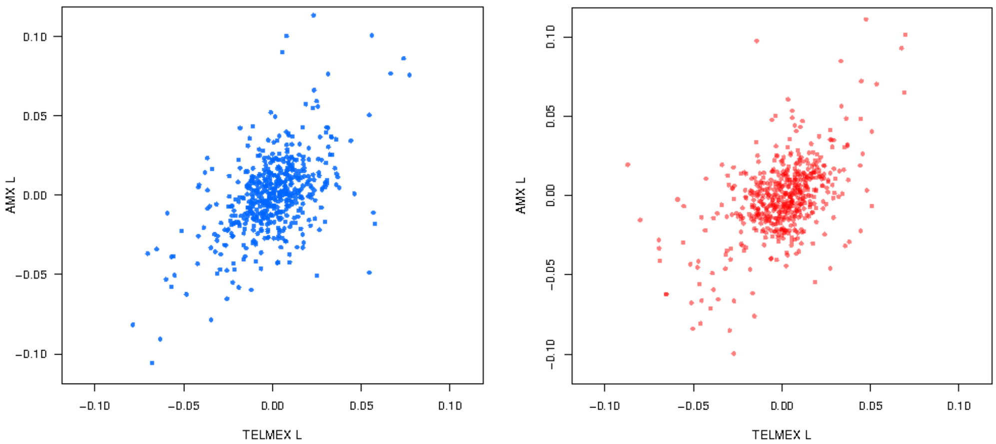

Figure 12. Scatter plots: (L) Observed sample, (R) Gaussian mixture simulation.

The Copula approach

Consider again our 529 random returns X and Y with marginal distributions F_X and F_Y respectively, and let $H : \mathbb{R}^2 \to \mathbb{R}$ be their corresponding cumulative joint distribution. According to Sklar's Theorem (see Nelsen [19], Chap. 2), there exists a copula function $C : [0,1] \times [0,1] \to \mathbb{R}$, such that

$$H(x,y) = C(F(x), G(y)) \qquad \forall (x,y) \in \mathbb{R}^2.$$

If F and G are continuous, then C is unique.

The Gaussian Copula model consists on the application of the bivariate normal as a copula function; this is: $C(u,v;\rho) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v)|\rho)$ where $\Phi(\cdot,\cdot|\rho)$ represents a bivariate normal cumulative distribution, centered at the origin, with unite variances and correlation coefficient ρ. If we let $u = F_X(x)$ and $v = F_Y(y)$ and we assume that the joint distribution F_{XY} of our assets returns is given in terms of a Gaussian copula, then

$$F_{xy}(x,y) = \Phi(\Phi^{-1}(F_X(x)), \Phi^{-1}(F_Y(y))|\rho) \qquad \forall (x,y) \in \mathbb{R}^2. \tag{10}$$

Its corresponding density is given by:

$$f_{XY}(x,y;\rho) = \frac{\phi(\Phi^{-1}(F_X(x)), \Phi^{-1}(F_Y(y))|\rho)}{\phi(\Phi^{-1}(F_X(x)))\phi(\Phi^{-1}(F_Y(y)))} f_X(x) f_Y(y). \tag{11}$$

In order to estimate the joint density of TELMEX L and AMX L, we applied a Gaussian copula with Gaussian mixtures as marginal densities for the observed sample of bivariate returns shown at the left hand side of Figure 12. The marginal distributions F_X and F_Y were considered to be the univariate GM models specified in Table 4. Given the joint density in equation (11), the parameter ρ was estimated by maximum likelihood:

$$\hat{\rho} = 0.4968.$$

A graphical assessment is shown in Figures 13 and 14. In general, the Gaussian copula model shows a good fit. Its curves of level are similar to those of the empirical density (Figure 13), and in a Monte Carlo simulation it is difficult to differentiate its scatter plot from the scatter of the observed sample (Figure 14).

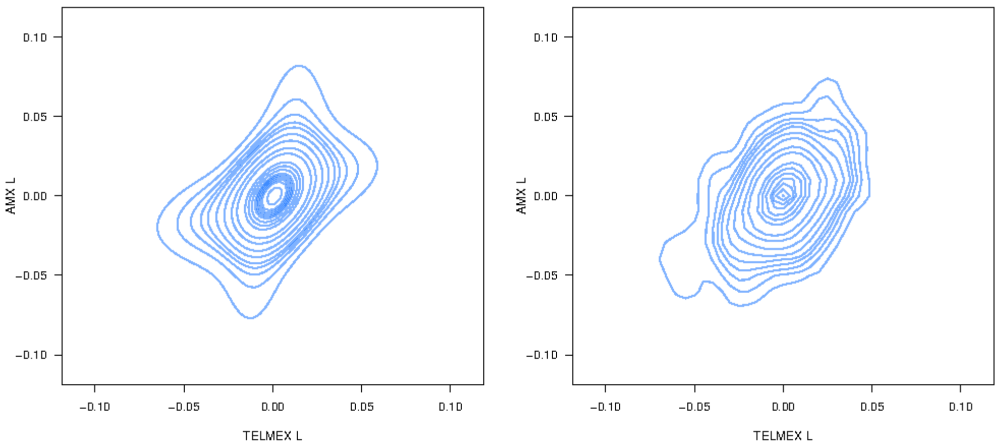

Figure 13. Curves of level: (L) Gaussian copula, (R) Smooth empirical density.

Comparative assessment of both models

The bivariate GM as well as the Gaussian copula with GM marginals seem to offer an acceptable fit. The plots shown in Figures 11 and 13 suggest that both functions are quite similar to the smooth empirical density. However, there are some clear differences between these two models. Both have elliptic curves of level in their high density regions near the mode. However, while the GM keeps this elliptical pattern constant, the curves of level of the Gaussian copula model gradually evolve into a rather irregular shape as the density decreases, adopting a form similar to the curves of level of the empirical density. In order to compare the quality of these two estimations, we decided to define a partition of the $\mathbb{R}^2$ space into nine subsets, and then to apply a Chi-square goodness of fit test.

This criterion is illustrated in Figure 15. The intervals on the X and Y axis are marginally equiprobable. The results of the test are given in Table 6. According to this Chi-Square statistic and to the previous graphical assessment, we can conclude that both models offer a good fit. Nevertheless, the bivariate GM seems to fit slightly better.

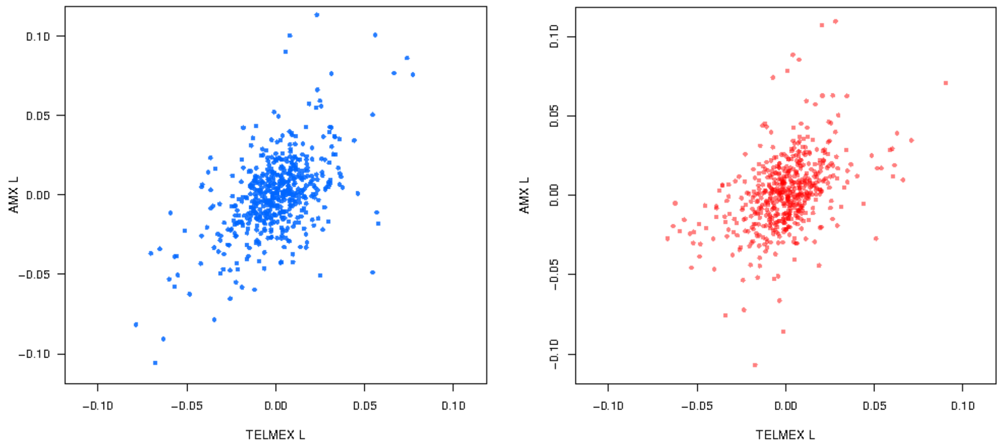

Figure 14. Scatter plots: (L) Observed sample, (R) Gaussian copula simulation.

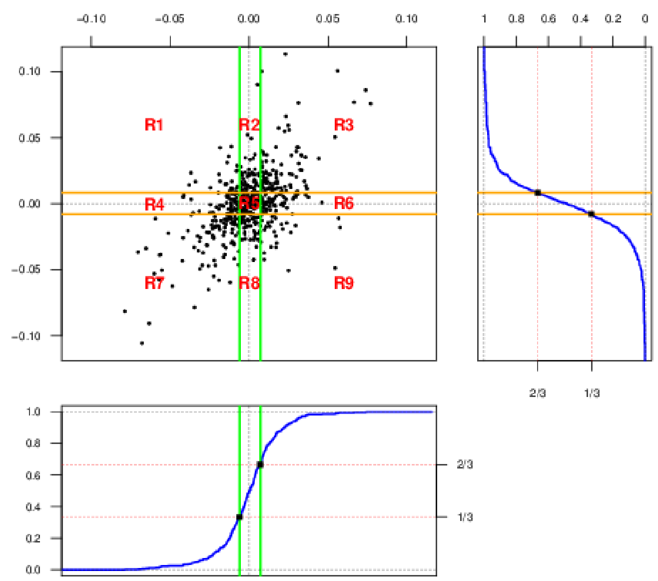

Figure 15. Space partition for a Chi-Square goodness of fit assessment.

Model	χ^2 Statistic	p-value
Bivariate Gaussian Mixture	2.0640	0.9790
Gaussian Copula with Gaussian Mixture Marginals	4.0997	0.8480

Table 6. Bivariate GM and GC with GM marginals. Goodness of fit assessment.

7. PORTFOLIO SELECTION AND RISK ASSESSMENT

Consider a random vector $X^t = (X_1, X_2, \ldots, X_n)$ distributed according to a mixture of k multivariate Normal densities, and let $S = \omega^t X = \sum_{i=1}^{p} \omega_i X_i$ be a convex linear combination of the elements of X. Then, S follows a univariate Gaussian Mixture.

To proof this, we apply the formula of the complete probability to express $F_S(s) = \Pr[S \leq s]$ as follows:

$$F_S(s) = \sum_{j=1}^{k} \pi_j \Pr[S \leq s \mid \Omega_j] = \sum_{j=1}^{k} \pi_j \Phi \left[\frac{s - \omega^t \mu_j}{\omega^t \Sigma_j \omega} \right], \tag{12}$$

where μ_j and Σ_j represent the mean vector and covariance matrix of the j^{th} element of the Gaussian Mixture model. The distribution of S expressed in equation (12) is a convex linear combination of k univariate normal distributions; therefore, S follows a univariate GM.

If X represents the vector of joint returns of p assets in a portfolio, then F_S is its corresponding distribution function. The Value at Risk (VaR) at level α for this portfolio is the solution to the following equation in s:

$$F_S(s) = \alpha.$$

Its corresponding Conditional Value at Risk (CVaR) is $\mathbb{E}(S \mid S \leq s)$.

We now consider the returns of TELMEX L and AMX L; both analyzed in the previous section. We create 100 different portfolios with weights ω and $1-\omega$, for $\omega = 0.00, 0.01, 0.02, \ldots, 0.99$. Figure (16) shows the VaR and CVaR at levels $\alpha = 0.01$ and $\alpha = 0.05$ for these hundred portfolios. Portfolios with optimal VaR where identified numerically. Their corresponding CVaR was estimated by Monte Carlo simulation. A numerical summary is shown in Table (7).

α	TELMEX L	AMX L	VaR	CVaR	Std-E
0.05	0.58	0.42	−1.55%	−2.5%	0.00005
0.01	0.59	0.41	−3.09%	−3.9%	0.00003

Table 7. VaR and CVaR for TELMEX L and AMX L

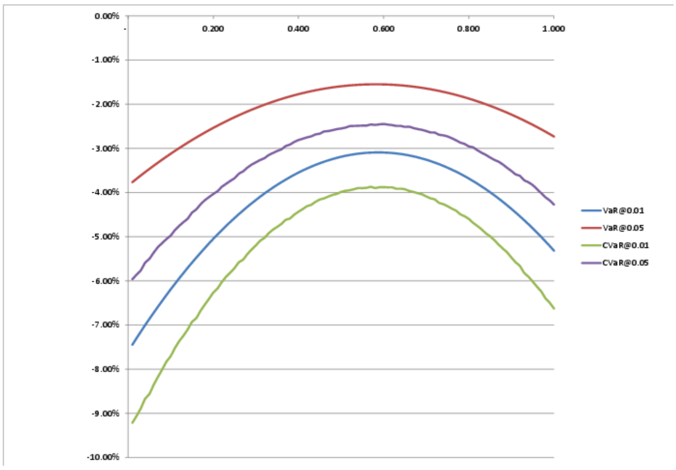

Figure 16. VaR at levels 0.01 and 0.05 for a hundred protfolios: TELMEX L and AMX L

8. Discussion and conclusion

Gaussian mixtures are flexible and easy to implement. To be based on the Normal distribution makes them an attractive and natural alternative in financial modeling.

Recent articles show successful applications of GM in different markets from Europe and Asia. Here, we illustrate their potential with three examples based on the Mexican market. The empirical evidence suggests that GM offer a powerful tool in financial modeling.

Gaussian mixtures may be leptokurtic but never heavy tailed. Section 5 shows three univariate examples where the application of GM models results in an accurate approximation to the actual distribution of three different series of returns. The goodness of fit assessment in these three examples is based on the Kolmogorov-Smirnov test. Therefore, it mainly concentrates on the body of the distribution while fit on the tails is almost ignored. If GM (or any other flexible model) are applied in practice, the additional application of the Anderson-Darling test together with the analysis of a qq-plot is strongly recommendable.

Section 6 presents the estimation of the joint density of two assets from the Mexican market. The analysis fits a bivariate GM and compares it to a Gaussian copula with GM marginals. The bivariate GM shows a slightly better fit. This is a direct consequence of a larger number of parameters in the bivariate GM. Both models (the mixture and the copula) show a very good fit. Nevertheless, it is our opinion that the GM model is more interpretable, and gives a better description of the joint behavior of both assets.

Finally, a simple exercise of portfolio selection and risk assessment based on a Gaussian Mixture model is presented in Section 7. This practical example illustrates the potential of multivariate GM models in financial practice. It shows how the GM approach simplifies the calculation of important risk measures such as VaR and CVaR. This idea was previously presented in Tand and Chu [20], however, their proposal implicitly assumes that the assets contained in the portfolio have independent returns. In contrast, our proposal takes into account the class conditional covariance structure of the multivariate GM model, resulting in a more realistic distributional assumption.

We have presented clear evidence that Gaussian mixtures constitute a competitive statistical tool with great practical value in financial modeling. They offer a versatile combination of precision and simplicity that makes them an interesting extension of the normal distribution in Quantitative Finance. In a univariate context, this parametric family shows a natural capacity to fit leptokurtic distributions. Three univariate examples illustrate how a parsimonious linear combination of just two normals may be enough to capture the leptokurtosis induced by random changes of volatility. In a bidimensional exercise, a mixture of two bivariate normals gave us an accurate estimate of the joint distribution of two assets returns. This bivariate model includes two correlation coefficients interacting on a random exchange of regimes process. Thus, it gives us a clear and interesting description of the dependence structure of both returns that is not so easily obtained from the Gaussian copula model. A quality assessment compares the bivariate GM with a Gaussian copula with GM marginals. Both models show a similar fit; however, the bivariate GM seems to be slightly more flexible and easier to interpret. Given their notorious flexibility, GM models constitute an interesting statistical tool in Quantitative Finance. Their relation to the normal distribution, invites us to study their implications in modern financial practice. We think that some fundamental concepts of Portfolio Optimization and Value at Risk deserve to be deeply studied from the finite mixture models perspective.

References

[1] A. Behr and U. Poetter, *Modeling Marginal Distributions of ten European Stock Market Index Returns*, International Research Journal of Finance and Economics **28** (2009) 104–118.

[2] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, The Journal of Political Economy **81** (1973) 637–654.

[3] I. Buckley, D. Saunders and L. Seco, *Portfolio optimization when asset returns have the Gaussian mixture distribution*, European Journal of Operational Research **185** (2008) 1434–1461.

[4] D. Esch, *Non-Normality facts and falacies*, Journal of Investment Management **8** (2010) 49–61.

[5] E.F. Fama, *The behavior of Stock Market Prices*, The Journal of Business **38** (1965) 34–105.

[6] S. Fruhwirth-Schnatter, Finite Mixture and Markov Switching Models (Springer, 2006).

[7] N.T. Gridgeman, *A comparison of two methods of analysis of mixtures of Normal distributions*, Technometrics **12** (1970) 823–833.

[8] J.C. Hull and A.D. White, *Value at risk when daily changes in market variables are not normally distributed*, The Journal of Derivatives **5** (1998) 9–19.

[9] E. Jondeau, S.H. Poon and M. Rockinger, Financial Modeling under non-Gaussian Distributions (Springer, 2000).

[10] Z.A. Kamaruzzaman, Z. Isa and M.T. Ismail, *Mixtures of Normal distributions, application to Bursa Malaysa stock market indices*, World Applied Sciences Journal **16** (2012) 781–790.

[11] S.A. Klugman, H.H. Panjer and G.E. Willmot, Loss Models, from data to decisions, (Wiley Series in Probability and Statistics, 2004).

[12] B. Mandelbrot, *The variation of certain speculative prices*, Journal of Business (1963) 394–419.

[13] K.V. Mardia, J.T. Kent and J.M. Bibby, Multivariate Analysis, (Academic Press, 1979).

[14] H. Markowitz, *Portfolio Selection*, The Journal of Finance **7** (1952) 77=-91.

[15] G.J. McLachlan, T. Krishnan and Ng. See Ket, *The EM Algorithm*, Papers / Humboldt-Universitat Berlin, Center for Applied Statistics and Economics (CASE) **24** (2004). Available at: http://hdl.handle.net/10419/22198

[16] G.J. McLachlan and T. Krishnan, The E.M. Algorithm and Extensions (Wiley Series in Probability and Statistics, 2008).

[17] G.J McLachlan and D. Peel, Finite Mixture models (Wiley Series in Probability and Statistics, 2000).

[18] P.P. Mota, *Normality Assumption for the log-return of the stock prices*, Discuss. Math. Probability and Statistics **32** (2012) 47–58.

[19] R.B. Nelsen, An introduction to Copulas (Springer, 1999).

[20] K. Tan and M. Chu, *Estimation of Portfolio Return and Value at Risk using a Class of Gaussian Mixture Distributions*, The International Journal of Business and Finance **6** (2012) 97–107.

[21] M.H. Zhang and Q.S. Cheng, *An approach to VaR for capital markets with Gaussian mixtures*, Applied Mathematics and Computation **168** (2005) 1079–1085.

Received 23 June 2017
Accepted 17 August 2017

Appendix

Proposal. Let $g : \mathbb{R} \to \mathbb{R}$ be a probability density function with the following features: g is symmetric, not platikurtic, centered around μ and it has finite variance σ^2. Then, the mixture of g with any normal density centered on the same μ as g but with different variance is leptokurtic.

Proof. It can be assumed, without loss of generality, that $\mu = 0$ and that g is mixed with a standard normal density. Thus, consider a random variable X which is distributed according to

$$f(x) = \pi\phi(x) + (1 - \pi)g(x).$$

The forth moment of X is $\mathrm{E}[X^4] = 3\alpha + (1 - \alpha)\mu_4$, where μ_4 is the fourth moment corresponding to g. In a similar way, $\mathrm{E}[X^2] = \alpha + (1 - \alpha)\sigma^2$. To prove this proposal is equivalent to solve the following inequality[2]:

$$\frac{3\pi + (1 - \pi)\mu_4}{(\pi + (1 - \pi)\sigma^2)^2} > 3.$$

The inequality above is true as long as

$$\pi\left(1 - \frac{1}{\sigma^2}\right)^2 + \frac{1}{3}\frac{\mu_4}{\sigma^4} > 1.$$

Given that $\pi(1 - \frac{1}{\sigma^2})^2 > 0$ and $\frac{1}{3}\frac{\mu_4}{\sigma^4} \geq 1$, we can conclude that the proposal is true. ■

Corollary. *The mixture of two or more normal densities with different variances, but with the same expected value μ is always leptokurtic.*

Proof. This corollary can be proofed by mathematical induction. Without loss of generality, it is assumed that $\mu = 0$. We know that normal densities are not platikurtic; therefore, according our previous proposal, the mixture of two normals densities is leptokurtic. Lets assume that the corollary is true for a mixture of $k > 2$ Gaussian densities, and consider a mixture of $k + 1$ normals with coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$. Let $\pi = (\sum_{j=1}^{k} \alpha_j)$ and $1 - \pi = \alpha_{k+1}$. Thus

$$\sum_{i=1}^{k+1} \frac{\alpha_i}{\sigma_i}\phi\big(x \,|\, \sigma_i^2\big) = \frac{1 - \pi}{\sigma_{k+1}}\phi\big(x \,|\, \sigma_{k+1}^2\big) + \pi\sum_{i=1}^{k}\left(\frac{\alpha_i}{\sum_{j=1}^{k}\alpha_j}\right)\frac{1}{\sigma_i}\phi\big(x \,|\, \sigma_i^2\big) \tag{13}$$

is the mixture of a non-platikurtic density (mixture of its first k elements) with a normal density ($(k + 1)^{\mathrm{th}}$ element of the mixture). Therefore, the corollary is true. ■

[2]The left hand side is the excess of kurtosis of f.